ON A SPECIAL CLASS OF SMOOTH CODIMENSION TWO SUBVARIETIES IN $\mathbb{P}^n$, $n \geq 5$

C. FOLEGATTI

1. Introduction

We work on an algebraically closed field of characteristic zero.
By Lefschetz's theorem, a smooth codimension two subvariety $X \subset \mathbb{P}^n$, $n \geq 4$, which is not a complete intersection, lying on a hypersurface Σ, verifies $dim(X \cap Sing(\Sigma)) \geq n-4$.
In this paper we deal with a situation in which the singular locus of Σ is as large as can be, but, at the same time, the simplest possible: we assume Σ is an hypersurface of degree m with an (m-2)-uple linear subspace of codimension two.
More generally, we are concerned with smooth codimension two subvarieties $X \subset \mathbb{P}^n$, $n \geq 5$.
In the first part we consider smooth subcanonical threefolds $X \subset \mathbb{P}^5$ and we prove that if $deg(X) \leq 25$, then X is a complete intersection (Prop. 2.2). In the second section we study a particular class of codimension two subvarieties and we prove the following result.

Theorem 1.1. *Let $X \subset \mathbb{P}^n$, $n \geq 5$, be a smooth codimension two subvariety (if $n = 5$ assume $Pic(X) = \mathbb{Z}H$) lying on a hypersurface Σ of degree m, which is singular, with multiplicity $m-2$, along a linear subspace K of dimension $n-2$. Then X is a complete intersection.*

This gives further evidence to Hartshorne conjecture in codimension two.
It is enough to prove the theorem for $n = 5$, the result for higher dimensions will follow by hyperplane sections. For $n = 5$ it is necessary to suppose $Pic(X) = \mathbb{Z}H$, whereas for $n \geq 6$, thanks to Barth's theorem, this hypothesis is always verified.
The proof for $n = 5$ goes as follows. Using the result of the first part we may assume $d \geq 26$, then we prove, under the special assumptions of the theorem, that either $deg(X)$ is less than 25 or we use the result of Lemma 3.3 to conclude that S is a complete intersection.
By the way we give a little improvement of earlier results on the non existence of rank two vector bundles on $\mathbb{P}^4$ with small Chern classes, see Lemma 2.8.
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2. Smooth subcanonical threefolds in $\mathbb{P}^5$

Let X be a smooth subcanonical threefold in $\mathbb{P}^5$, of degree d, with $\omega_X \cong \mathcal{O}_X(e)$. Let $S = X \cap H$ be the general hyperplane section of X, S is a smooth subcanonical surface in $\mathbb{P}^4$, indeed by adjunction it is easy to see that $\omega_S \cong \mathcal{O}_S(e+1)$. Again we set C the general hyperplane section of S, C is a smooth subcanonical curve in $\mathbb{P}^3$, with $\omega_C \cong \mathcal{O}_C(e+2)$.
We can compute the sectional genus $\pi(S)$, indeed since $\omega_C \cong \mathcal{O}_C(e+2)$ it follows that $\pi = g(C) = 1 + \frac{d(e+2)}{2}$.

Lemma 2.1. *With the notations above, $q(S) = 0$ and all hyperplane sections C of S are linearly normal in $\mathbb{P}^3$.*

Proof: By Barth's theorem we know that if $X \subset \mathbb{P}^5$ is a smooth threefold, then $h^1(\mathcal{O}_X) = 0$. Let us consider the exact sequence: $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$. By taking cohomology and observing that $h^2(\mathcal{O}_X(-1)) = h^1(\omega_X(1)) = 0$ by Kodaira, we get the result.$\diamondsuit$

If we look at the surface S, we can observe that most of its invariants are known. Hence it seems natural to consider the double points formula in order to get some more information.
Since $q(S) = 0$, $\pi - 1 = \frac{d(e+2)}{2}$ and $K^2 = d(e+1)^2$, the formula becomes: $d(d - 2e^2 - 9e - 17) = -12(1 + p_g(S))$, where the quantity $1 + p_g(S)$ is strictly positive. We have the following condition:

$$d(d - 2e^2 - 9e - 17) \equiv 0 \ (mod\ 12) \qquad (1)$$

Proposition 2.2. *Let $X \subset \mathbb{P}^5$ be a smooth subcanonical threefold of degree d, then if $d \leq 25$, X is a complete intersection.*

Proof: We recall that for a smooth subcanonical threefold in $\mathbb{P}^5$ with $\omega_X \cong \mathcal{O}_X(e)$ we have $e \geq 3$, unless X is a complete intersection (see [1]). Let $G(d,3) = 1 + \frac{d(d-3)-2r(3-r)}{6}$ be the maximal genus of a curve of $\mathbb{P}^3$ of degree $d = 3k+r$, $0 \leq r \leq 2$, not lying on a surface of degree two. If we compare the value of π computed before with this (using $e \geq 3$), we see that if $d \leq 17$, then $h^0(\mathcal{I}_C(2)) \neq 0$. Since by Severi's and Zak's theorems on linear normality $h^1(\mathcal{I}_S(1)) = h^1(\mathcal{I}_X(1)) = 0$, it follows that $h^0(\mathcal{I}_X(2)) \neq 0$ and this implies that X is a complete intersection (see [2], Theorem 1.1).
If $d = 18$, then $\pi = G(18,3)$. It follows that C is a.C.M. then by the exact sequence: $0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{E} \to \mathcal{I}_C(e+6) \to 0$ we obtain $h^1(\mathcal{E}(k)) = 0$ for all $k \in \mathbb{Z}$. Hence by Horrocks' theorem $\mathcal{E}$ is split and then C is a complete intersection. Since this holds

for the general $\mathbb{P}^3$ section C, the same holds for S and for X.
If $d = 19$ then C lies on a quadric surface unless $\pi = 1 + \frac{19(e+2)}{2} \leq G(19,3)$. This inequality yields $e = 3$ but if we look at formula (1) we see that this is not possible.
If $d = 20$, then $\pi = G(20,4)$, C is a.C.M. and we argue as in the case $d = 18$ to conclude that X is a complete intersection.
If $d = 21, 22, 23$ and if $h^0(\mathcal{I}_C(4)) \neq 0$, then thanks to the "lifting theorems" in $\mathbb{P}^4$ and $\mathbb{P}^5$ (see [11]) we have $h^0(\mathcal{I}_X(4)) \neq 0$ and again by [2] X is a complete intersection. We then assume $h^0(\mathcal{I}_C(4)) = 0$ and using the fact that $\pi = 1 + \frac{d(e+2)}{2} \leq G(d,5)$, we obtain $e = 3$. However this is not possible because of formula (1).
If $d = 24$ we still get $e = 3$, but formula (1) is satisfied. We have the following exact sequence $0 \to \mathcal{O} \to \mathcal{E}(5) \to \mathcal{I}_X(9) \to 0$, where $\mathcal{E}$ is a rank two vector bundle with $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 4$. If $h^0(\mathcal{I}_X(4)) = 0$, then $h^0(\mathcal{E}) = 0$, which is not possible since by [4] there exists no rank two stable vector bundle with such Chern classes. Hence it would be $h^0(\mathcal{I}_X(4)) \neq 0$ and this implies (see [2], Theorem 1.1) that X is a complete intersection but this is also impossible since the system given by the equations $a + b = -1$ and $ab = 4$ does not have solution in $\mathbb{Z}$.
If $d = 25$, supposing $h^0(\mathcal{I}_C(4)) = 0$ we obtain $e = 4$. In that case we have exactly $\pi = G(25,5) = 76$ and this means that if $h^0(\mathcal{I}_C(4)) = 0$, then C is a.C.M.. It follows that C, and then X, is a complete intersection. $\diamondsuit$

Remark 2.3. If we perform the same calculations of the proof of 2.2 for $d = 26$, we have that $e = 3$.
Now if we consider subcanonical threefolds in $\mathbb{P}^5$ with $e = 3$, by Kodaira we have that $h^0(\mathcal{O}_X(4)) = \chi(\mathcal{O}_X(4))$. By Riemann-Roch formula for threefolds (see [1]) we compute $\chi(\mathcal{O}_X(4)) = \frac{5d(50-d)}{24}$. Since $h^0(\mathcal{O}_{\mathbb{P}^5}(4)) = 126$, it is easy to see that for $d \geq 30$ it must be $h^0(\mathcal{I}_X(4)) \neq 0$, hence X is a complete intersection.
On the other hand for $26 \leq d \leq 30$ the unique value of d satisfying (1) is $d = 26$. Thus we have shown that, among smooth threefolds in $\mathbb{P}^5$ with $e = 3$, the only possibility for X not to be a c.i. is if $d = 26$.

We conclude this section with some result about rank two vector bundles. Let us start with a lemma concerning subcanonical double structures.

Lemma 2.4. *Let $Y \subset \mathbb{P}^n$, $n \geq 4$, be a complete intersection of codimension two. Let Z be a l.c.i. subcanonical double structure on Y. Then if $emdim(Y) \leq n-1$, Z is a complete intersection.*

Proof: By [10] we have that any doubling of a l.c.i. Y with $emdim(Y) \leq dim(Y) + 1$ is obtained by the Ferrand construction. Hence there is a surjection $\mathcal{N}_Y^\vee \to \mathcal{L} \to 0$ where $\mathcal{L}$ is a locally free sheaf of rank one on Y. Taking into account that $\omega_{Z|Y} \cong \omega_Y \otimes \mathcal{L}^\vee$ (see [8]) and recalling that Z is subcanonical and Y is a c.i., we obtain that $\mathcal{L} \cong \mathcal{O}_Y(l)$ for a certain $l \in \mathbb{Z}$.

On the other hand, since Y is a complete intersection, say $Y = F_a \cap F_b$, we have $\mathcal{N}_Y \cong \mathcal{O}_Y(a) \oplus \mathcal{O}_Y(b)$, then the sequence above becomes: $\mathcal{O}_Y(-a) \oplus \mathcal{O}_Y(-b) \xrightarrow{f} \mathcal{O}_Y(l) \to 0$. The map f is given by two polynomials of degree respectively $a+l$ and $b+l$. If F and G are both not constant, it follows, since $n \geq 4$, that $B := (F)_0 \cap (G)_0 \cap Y \neq \emptyset$. For each $x \in B$ the induced map f_x on the stalks is not surjective: absurd. Thus necessarily F or G is a non zero constant, i.e. either $l = -a$ or $l = -b$. If $l = -a$ (resp. $l = -b$) we are doubling Y on F_b, $Z = F_a^2 \cap F_b$ (resp. we are doubling Y on F_a, $Z = F_a \cap F_b^2$). In any case, Z is a complete intersection.$\diamondsuit$

Lemma 2.5. *Let $Z \subset \mathbb{P}^4$ be a l.c.i. quartic surface with $\omega_Z \cong \mathcal{O}_Z(-a)$. If $a \geq 3$, then Z is a complete intersection.*

Proof: Let C be the hyperplane section of Z and let $C_{red} = \tilde{C}_1 \cup \ldots \cup \tilde{C}_s$ be the decomposition of C_{red} in irreducible components, hence $C = C_1 \cup \ldots \cup C_s$, where C_i is a multiple structure on $\tilde{C}_i$ for all i . We have $\omega_C \cong \mathcal{O}_C(-a+1)$, on the other hand $\omega_{C|C_i} \cong \omega_{C_i}(\Delta)$, where Δ is the scheme theoretic intersection of C_i and $\bigcup_{i\neq j} C_j$. It follows that $\omega_{C_i} \cong \mathcal{O}_{C_i}(-a+1-\Delta)$ and since $deg(\Delta) \geq 0$, this implies that $p_a(C_i) < 0$, then C_i is a multiple structure on $\tilde{C}_i$ of multiplicity > 1.
It turns out that each irreducible component of Z_{red} appears with multiplicity > 1, thus since $deg(Z) = 4$ it follows that Z is a double structure on a quadric surface or a 4-uple structure on a plane. This last case can be readily solved. Indeed C would be a 4-uple structure on a line and thanks to [8] (Remark 4.4) we know that a thick and l.c.i. 4-uple structure on a line is a global complete intersection. Hence we can assume Z quasi-primitive, i.e. we can assume Z does not contain the first infinitesimal neighbourhood of Z_{red}. Anyway by [9] (see main theorem and Section B) and since Z_{red} is a plane we also have that Z is a c.i..
We then suppose that Z is a double structure on a quadric surface of rank ≥ 2, which is a complete intersection $(1,2)$. By Lemma 2.4 it follows that Z is a c.i..$\diamondsuit$

Definition 2.6. Let $\mathcal{E}$ be a rank two normalized vector bundle (i.e. $c_1(\mathcal{E}) = -1, 0$), we set $r := min\{n | h^0(\mathcal{E}(n)) \neq 0\}$. If $r > 0$, $\mathcal{E}$ is stable. If $r \leq 0$ we call r *degree of instability of* $\mathcal{E}$.

Remark 2.7. The next lemma represents a slight improvement of previous results about the existence of rank two vector bundles in $\mathbb{P}^4$ and $\mathbb{P}^5$.
Indeed Decker proved that any stable rank two vector bundle on $\mathbb{P}^4$ with $c_1 = -1$ and $c_2 = 4$ is isomorphic to the Horrocks-Mumford bundle and that in $\mathbb{P}^5$ there is no stable rank two vector bundle with these Chern classes (see [4]). We show that neither are there such vector bundles with $r = 0$. As for bundles with $c_1 = 0$ and $c_2 = 3$, there are similar results by Barth-Elencwajg (see [5]) and Ballico-Chiantini (see [1]) stating that $r < 0$. We prove that in fact $r < -1$.

Lemma 2.8. *There does not exist any rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^4$ such that $r = 0$, $c_1(\mathcal{E}) = -1$, $c_2(\mathcal{E}) = 4$ or, respectively, $r = -1$, $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 3$.*

Proof: We observe first of all that in both cases there are no integers a, b satisfying the equations $a + b = c_1$, $ab = c_2$, hence the vector bundle $\mathcal{E}$ cannot be split.
Assume $\mathcal{E}$ has $r = 0$, $c_1(\mathcal{E}) = -1$, $c_2(\mathcal{E}) = 4$, then $h^0(\mathcal{E}) \neq 0$. There is a section of $\mathcal{E}$ vanishing on a codimension two scheme Z: $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_Z(-1) \to 0$. We have $deg(Z) = c_2(\mathcal{E}) = 4$ and Z subcanonical with $\omega_Z \cong \mathcal{O}_Z(-6)$.
If $r = -1$, $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 3$, then $h^0(\mathcal{E}(-1)) \neq 0$ and we get a section of $\mathcal{E}(-1)$ vanshing in codimension two along a quartic surface Z, with $\omega_Z \cong \mathcal{O}_Z(-7)$.
It is enough to apply 2.5 to conclude that such vector bundles cannot exist.$\diamondsuit$

3. Codimension two subvarieties in $\mathbb{P}^n$, $n \geq 5$

Let $X \subset \mathbb{P}^n$, $n \geq 5$ be a smooth codimension two subvariety, lying on a hypersurface Σ of degree $m \geq 5$ with a (m-2)-uple linear subspace K of codimension two, i.e. $K \cong \mathbb{P}^{n-2}$. If $n = 5$ we assume $Pic(X) = \mathbb{Z}H$, for $n \geq 6$ this is granted by Barth's theorem. In any case we set $\omega_X \cong \mathcal{O}_X(e)$.
The general $\mathbb{P}^4$ section S of X is a surface lying on a threefold $\Sigma \cap H$ of degree m having a singular plane of multiplicity (m-2). We will always suppose that $h^0(\mathcal{I}_S(2)) = 0$.
We will prove that S contains a plane curve. First we fix some notations and state some results concerning surfaces containing a plane curve, proofs and more details can be found in [3].
Let P be a plane curve of degree p, lying on a smooth surface $S \subset \mathbb{P}^4$. Let Π be the plane containing P and let $Z := S \cap \Pi$. We assume that P is the one-dimensional part of Z and we define $\mathcal{R}$ as the residual scheme of Z with respect to P, namely $\mathcal{I}_{\mathcal{R}} := (\mathcal{I}_Z : \mathcal{I}_P)$. The points of the zero-dimensional scheme $\mathcal{R}$ can be isolated as well as embedded in P.
Let δ be the ∞^1 linear system cut out on S, residually to P, by the hyperplanes containing Π. Severi's theorem states that unless S is a Veronese surface, then $h^1(\mathcal{I}_S(1)) = 0$ and thus $H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \cong H^0(\mathcal{O}_S(1))$. Moreover if $p \geq 2$, the hyperplanes containing Π are exactly those containing P. This allows us to conclude that $\delta = |H - P|$ (on S). We will denote by Y_H the element of δ corresponding to the hyperplane H and we call $C_H = P \cup Y_H = S \cap H$.
Let $\mathcal{B}$ be the base locus of δ. We have the following results.

Lemma 3.1. *(i) P is reduced, the base locus $\mathcal{B}$ is contained in Π and $dim(\mathcal{B}) \leq 0$. The general $Y_H \in \delta$ is smooth out of Π and does not have any component in Π.*
(ii) $\mathcal{B} = \mathcal{R}$ and $\deg(\mathcal{R}) = (H - P)^2 = d - 2p + P^2$.

Proof: See Lemma 2.1 and 2.4 of [3].$\diamondsuit$

In the present situation, S is subcanonical with $\omega_S \cong \mathcal{O}_S(e+n-4)$. We know $deg(\mathcal{R}) = d - 2p + P^2$ and we compute P^2 by adjunction, knowing $p_a(P)$ since P is a plane curve and recalling that $K_S = (e+n-4)H$. It turns out that $deg(\mathcal{R}) = d + p^2 - p(e+n+1)$.

Lemma 3.2. *If $S \subset \mathbb{P}^4$ is a smooth surface, lying on a degree m hypersurface Σ with a $(m-2)$-uple plane, then S contains a (reduced) plane curve, P. If H is a general hyperplane through P, then $H \cap S = P \cup Y_H$ where Y_H has no irreducible components in Π and is smooth out of Π.*

Proof: If Π is the plane with multiplicity $(m-2)$ in Σ and H is an hyperplane containing Π, we have $H \cap \Sigma = (m-2)\Pi \cup Q_H$, where Q_H is a quadric surface and $C_H = S \cap H \subset (m-2)\Pi \cup Q_H$. If $dim(C_H \cap \Pi) = 0$, then $C_H \subset Q_H$, but this is excluded by our assumptions. Indeed by Severi's theorem $h^0(\mathcal{I}_{C_H}(2)) \neq 0$ would imply $h^0(\mathcal{I}_S(2)) \neq 0$. So $dim(C_H \cap \Pi) = 1$ and S contains a plane curve. We conclude with Lemma 3.1. $\diamondsuit$

If H is an hyperplane through Π, the corresponding section is $C_H = Y_H \cup P$. Since Y_H does not have any component in Π, we have $Y_H \subset Q_H$. We denote by q_H the conic $Q_H \cap \Pi$. As H varies, the q_H's form a family of conics in Π. Let $\mathcal{B}_q$ be the base locus of $\{q_H\}$, we have $\mathcal{R} \subset \mathcal{B}_q$, since $Y_H \cap \Pi \subset Q_H \cap \Pi = q_H$.
One can show that $\mathcal{B}_q$ is $(m-1)$-uple in Σ (see [3], Lemma 3.3). To prove this, just consider an equation φ of Σ and note that clearly $\varphi \in \mathbb{I}^2(\Pi)$. Easy computations show that all $(s-2)$-th derivatives of φ vanish at a point $x \in \mathcal{B}_q$.
The following result concerns in particular subcanonical surfaces.

Lemma 3.3. *With notations as above (S subcanonical with $\omega_S \cong \mathcal{O}_S(a)$), we have:*
(i) $deg(P) \leq a+3$.
(ii) If $\mathcal{R} = \emptyset$, then S is a complete intersection.

Proof: (i) We have already computed $deg(\mathcal{R}) = -p(a+5) + d + p^2$. Recall that $deg(Y_H) = d - p$ and $deg(\mathcal{R}) \leq deg(Y_H)$, this implies $p \leq a+4$. We will see that the case $p = a+4$ is not possible. Let $p = a+4$, then $Y_H \cdot P = p - P^2 = -p(p-a-4) = 0$, i.e. $Y_H \cap P = \emptyset$. In other words the curve $C_H = S \cap H = Y_H \cup P$ is not connected, but this is impossible since $h^0(\mathcal{O}_{C_H}) = 1$ (use $0 \to \mathcal{O}_S(-1) \to \mathcal{O}_S \to \mathcal{O}_{C_H} \to 0$ and $h^1(\mathcal{O}_S(-1)) = h^1(\omega_S(1)) = 0$ by Kodaira).
(ii) Since S is subcanonical we can consider the exact sequence $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_S(a+5) \to 0$. If we restrict it to Π and divide by an equation of P, we get $0 \to \mathcal{O}_\Pi \to \mathcal{E}_\Pi(-p) \to \mathcal{I}_\mathcal{R}(a+5-2p) \to 0$. If $\mathcal{R} = \emptyset$, then $\mathcal{I}_\mathcal{R} = \mathcal{O}_\Pi$ and the above sequence splits. It follows that $\mathcal{E}$ splits and S is a complete intersection. $\diamondsuit$

Example 3.4. Let S be a smooth section of the Horrocks-Mumford bundle $\mathcal{F}$, S is an abelian variety and has $\omega_S = \mathcal{O}_S$. By Lemma 3.3 we know that if S contains

a plane curve P, then $p \leq 3$. Moreover, P cannot be a line or a conic, since these curves are rational and this would imply that there exists a non constant morphism $\mathbb{P}^1 \to S$, factoring through $Jac^0(\mathbb{P}^1) \cong \{*\}$ and this is not possible. Then necessarily P is a plane smooth cubic (hence elliptic).
By the "reducibility lemma" of Poincaré, an abelian surface S contains an elliptic curve if and only if S is isogenous to a product of elliptic curves. It is known that the general section of the Horrocks-Mumford bundle is not isogenous to a product of elliptic curves, but there exist smooth sections satisfying such property (see [7], [6]). Summarizing we can say that among the sections of Horrocks-Mumford bundle we can find smooth surfaces containing a plane curve, but the general one does not contain any.
Now assume S to be one of those smooth surfaces containing a plane cubic, P. Let Π be the plane spanned by P. Recall that we have $0 \to \mathcal{O} \xrightarrow{s} \mathcal{F}(3) \to \mathcal{I}_S(5) \to 0$. We restrict the sequence to Π and since $s_{|\Pi}$ vanishes along P, we can divide by an equation of P and obtain a section of $\mathcal{F}_{|\Pi}$. We then have $h^0(\mathcal{F}_{|\Pi}) \neq 0$, i.e. $\mathcal{F}_{|\Pi}$ is not stable, in other words Π is an unstable plane for $\mathcal{F}$.

In order to prove Theorem 1.1 we need some other preliminary results.

Lemma 3.5. *Let $F \subset \mathbb{P}^3$ be a surface of degree m, singular along a line D with multiplicity $m-1$. Then F is the projection of a surface of degree m in $\mathbb{P}^{m+1}$ (minimal degree surface).*

Proof: The surface F is rational. Let $p : F' \to F$ be a desingularization of F and let H be a divisor in $p^*\mathcal{O}_F(1)$. We have $0 \to \mathcal{O}_{F'} \to \mathcal{O}_{F'}(H) \to \mathcal{O}_H(H) \to 0$ and since F' is rational too, then $h^1(\mathcal{O}_{F'}) = 0$. Now $h^0(\mathcal{O}_H(H)) = m+1$ (H is a rational curve), then $h^0(\mathcal{O}_{F'}(H)) = m+2$ and we can embed F' in $\mathbb{P}^{m+1}$.$\diamondsuit$

Remark 3.6. Minimal degree surfaces in $\mathbb{P}^n$ are classified, in particular they can be: a smooth rational scroll, a cone over a rational normal curve of $\mathbb{P}^{n-1}$ or the Veronese surface if $n = 5$. Except for the Veronese, all these surfaces are ruled in lines.

Lemma 3.7. *Let $T \subset \mathbb{P}^{m+1}$, $m \geq 3$, be a surface ruled in lines. Let $C \subset T$ be a smooth irreducible curve. If $dim(< C >) = 3$, then $deg(C) \leq deg(T) - m + 3$. ($< C >$ is the linear space spanned by C)*

Proof: Let us consider $m-3$ general points on C and let $f_1, \dots, f_{m-3}$ be the rulings passing through these points. We consider moreover $m-3$ points $p_1, \dots, p_{m-3}$ such that $p_i \in f_i$ but $p_i \notin < C >$ and let also $q_1, \dots, q_4$ be four general points in $< C >$. We thus have $m+1$ points, spanning at most a space of dimension m, hence these points are contained in a hyperplane H of $\mathbb{P}^{m+1}$. Now $< C > \subset H$ since $q_i \in H \ \forall\ i = 1, \dots, 4$, $f_i \subset H$ since $card(f_i \cap H) > 1 \ \forall\ i = 1, \dots, m-3$, so $H \cap T$ contains $C, f_1, \dots, f_{m-3}$ (which form a degenerate curve in T of degree

$m-3+deg(C))$ and this yields: $deg(T) \geq deg(C)+m-3.\diamondsuit$

Lemma 3.8. *Let $X, K \subset \mathbb{P}^n$, $n \geq 4$, X smooth of codimension two, $K \cong \mathbb{P}^{n-2}$ a linear subspace. Let $dim(X \cap K) = n-3$. If the general hyperplane section of $X \cap K$ contains a linear subspace of dimension $n-4$, then X contains a linear subspace of dimension $n-3$.*

Proof: We see $X_K = X \cap K$ as a hypersurface in $K \cong \mathbb{P}^{n-2}$. A general hyperplane of K is cut on K by a general hyperplane of $\mathbb{P}^n$. Then the hypersurface X_K of K is such that its general hyperplane section contains a linear subspace of dimension $n-4$. We claim that X_K contains an hyperplane of K. Indeed we may assume X_K reduced. Let $X_K = T_1 \cup \ldots \cup T_r$ be the decomposition of X_K into irreducible components. Now using the fact that the general hyperplane section of each T_i is irreducible, we conclude that one of the T_i's has degree one and thus X_K contains an hyperplane of $K.\diamondsuit$

Proof of Theorem 1.1: We only need to work out the case $n=5$. We will follow the method used in the proof of Theorems 1.1 and 1.2 of [3]. We must distinguish three cases, depending on the behaviour of the base locus $\mathcal{B}_q$ of the conics q_H. If $dim(\mathcal{B}_q) = 0$, at least two of the conics intersect properly and then $deg(\mathcal{B}_q) \leq 4$. It follows that $r := deg(\mathcal{R}) \leq 4$ too, since $\mathcal{R} \subset \mathcal{B}_q$. If $dim(\mathcal{B}_q) = 1$, there are two possibilities: the one-dimensional part of $\mathcal{B}_q$ can be a line or a conic.
If the conics q_H move, i.e. if $dim(\mathcal{B}_q) = 0$, we have seen that $r = deg(\mathcal{R}) \leq 4$. We observe that actually we can suppose $r \geq 1$, indeed by 3.3 if $\mathcal{R} = \emptyset$, then S (and X) is a complete intersection.
If H is a general hyperplane, $Y_H \cap P \subset q_H \cap P$ and since at least one conic intersects P properly, we obtain $Y_H.P \leq 2p$. We have $Y_H.P = p - P^2$ and recalling that $r = d - 2p + P^2$, it follows $Y_H.P = d-p-r$. Putting everything together: $p \geq \frac{d-r}{3} \geq \frac{d-4}{3}$. On the other hand we have $Y_H.P = p(e+5-p)$ and clearly this implies $p \geq e+3$. Comparing this with the result stated in 3.3 and setting $\omega_S \cong \mathcal{O}_S(e+1)$, we are left with only two possibilities: $p = e+3$ or $p = e+4$. We have already observed that $d = p(e+6) - p^2 + r$, then considering the two cases above, we can express d in terms of e and r and we get the following formulas:

$$if\ \ p = e+3,\ then\ d = 3(e+3)+r \qquad (2)$$

$$if\ \ p = e+4,\ then\ d = 2(e+4)+r \qquad (3)$$

We recall that if C lies on a quartic surface and d is large enough, X lies on a quartic hypersurface too, then X is a complete intersection. We know that $\pi - 1 = \frac{d(e+2)}{2}$, then since $\frac{d-4}{3} \leq p \leq e+4$ we obtain $\pi - 1 \geq \frac{d(d-10)}{6}$. If we compare this quantity with $G(d,5)$, we see that if $d \geq 33$, then $h^0(\mathcal{I}_C(4)) \neq 0$ and X is a complete intersection.

Thanks to the result in Proposition 2.2 we know that if $d \leq 25$, X is a complete intersection too, then we only have to check the cases $26 \leq d \leq 32$.
We assume $h^0(\mathcal{I}_C(4)) = 0$, then it must be $\pi = 1 + \frac{d(e+2)}{2} \leq G(d,5)$. Thanks to this inequality it is easy to see that for $d \leq 32$, we always have $e \leq 5$. Now if we look at formulas (2) and (3) above, clearly $e \leq 5$ implies $d \leq 28$.
On the other hand, in order to have $d \geq 26$, e must be at least equal to 4.
If $d = 26, 27, 28$, the condition on the genus π yields $e = 4$ again. However, if we look at formulas (2) and (3) we see that if $e = 4$, d is at most equal to 25.
If $\mathcal{B}_q$ contains a line, D, then D has multiplicity $m-1$ in Σ, so if H is an hyperplane containing D (but not Π), $F = \Sigma \cap H$ is a surface of degree m in $\mathbb{P}^3$ having a $(m-1)$-uple line. This kind of surface is a projection of a degree m surface in $\mathbb{P}^{m+1}$, by Lemma 3.5. The hyperplane section $C = S \cap H$ is a curve contained in F. We must distinguish two cases: $D \subset S$ or $D \not\subset S$.
If $D \not\subset S$, we claim that the general C is smooth. Let $|L|$ be the linear system cut on S by the hyperplanes containing D and let $B = D \cap S = \{p_1, \ldots, p_r\}$. Since B is the base locus of $|L|$, the general element of $|L|$ is smooth out of B. If all curves in $|L|$ were singular at a point $p_i \in B$, it would be $T_{p_i}S \subset H$, $\forall H \supset D$. Anyway the intersection of $H \supset D$ is only D, so this is not possible. It follows that the curves of $|L|$ singular at a $p_i \in B$ form a closed subset of $|L|$. The same holds for all $p \in B$, hence the claim.
Let F' be a surface in $\mathbb{P}^{m+1}$ projecting down to F. Since C is not contained in the singular locus of F, there exists a curve $C' \subset F'$ such that the projection restricted to C' is an isomorphism over C. In particular $\mathcal{O}_{C'}(1) \cong \mathcal{O}_C(1)$ and since C is linearly normal in $\mathbb{P}^3$, this implies that C' is degenerate. Now we can apply Lemma 3.7 to F' and C' (we have already pointed out that F' is ruled in lines unless F' is the Veronese surface) and we get $d = deg(C') \leq m - m + 3 = 3$. If F' is the Veronese surface we have anyway $d \leq 4$.
If $D \subset S$, then D is a component of the plane curve P (the one-dimensional part of $S \cap \Pi$). Coming back to the variety $X \subset \Sigma \subset \mathbb{P}^5$ with $K \cong \mathbb{P}^3 \subset \Sigma$ a linear subspace of multiplicity $m-2$, we have a surface $X_K = X \cap K \subset K \cong \mathbb{P}^3$ such that its general hyperplane section contains a line. This implies by Lemma 3.8 that X_K contains a plane and thus X contains a plane, say E. This plane is a Cartier divisor on the smooth threefold X. Since we are supposing $Pic(X) = \mathbb{Z}H$, there exists an hypersurface such that E is cut on X by this hypersurface, but this could happen only if $deg(X) = 1$.
To complete the proof we only have to consider the case in which $\mathcal{B}_q = q$, where q is an irreducible conic (if q is reducible, $\mathcal{B}_q$ contains a line). For every $Y_H \in |H - P|$ we consider the zero-dimensional scheme $\Delta_H = Y_H \cap q$. For every H, Δ_H is a subset of $d - p$ points of q.
There are two possibilities: $q \subset S$ or $q \not\subset S$. If $q \not\subset S$, then Δ_H is fixed (otherwise

the points of Δ_H would cover the conic, as H varies, i.e. $q \subset S$). It must be $\Delta_H = \mathcal{R}$. It is enough to compare the degrees of Δ_H and $\mathcal{R}$ to see that this implies $P^2 = p$ and then $Y_H.P = P^2 - p = 0$. This is not possible since the corresponding hyperplane section C_H of S would be disconnected.
Hence $q \subset S$ and then $q \subset P$. In other words: $\Delta_H = Y_H \cap P$, thus $Y_H P = d - p$ and $r = 0$. By Lemma 3.3 we conclude that X is a complete intersection.$\diamondsuit$

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Address of the author:
Dipartimento di Matematica
35, via Machiavelli
44100 Ferrara (Italy)
email: chiara@dm.unife.it